Interpreting Galilean Invariant Vector Field Analysis via Extended Robustness: Extended Abstract

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Motivation. Understanding vector fields is integral to many scientific applications ranging from combustion to global oceanic eddy simulations. Critical points of a vector field (zeros of the field) are essential features of the data, and play an important role in describing and interpreting the flow behavior. However, vector field analysis based on critical points suffers a major drawback: the definition of critical points depends upon the chosen frame of reference. Fig. 1 highlights this limitation, where the critical points in a simulated flow (the von Kármán vortex street) are only visible when the velocity of the incoming flow is subtracted.

The extraction of *meaningful* features in the data therefore depends on a *good* choice of a reference frame. Often times there exists no single frame of reference that enables simultaneous visualization of *all* relevant features. For example, it is not possible to find one single frame that simultaneously shows the von Kármán vortex street from Fig. 1(b), and the first vortex formed directly behind the obstacle in Fig. 1(a). To overcome such a drawback, a framework recently introduced by Bujack et al. [1] considers every point as a critical point and locally adjusts the frame of reference to enable simultaneous visualization of dominating frames that highlight features of interest. Such a framework selects a subset of critical points

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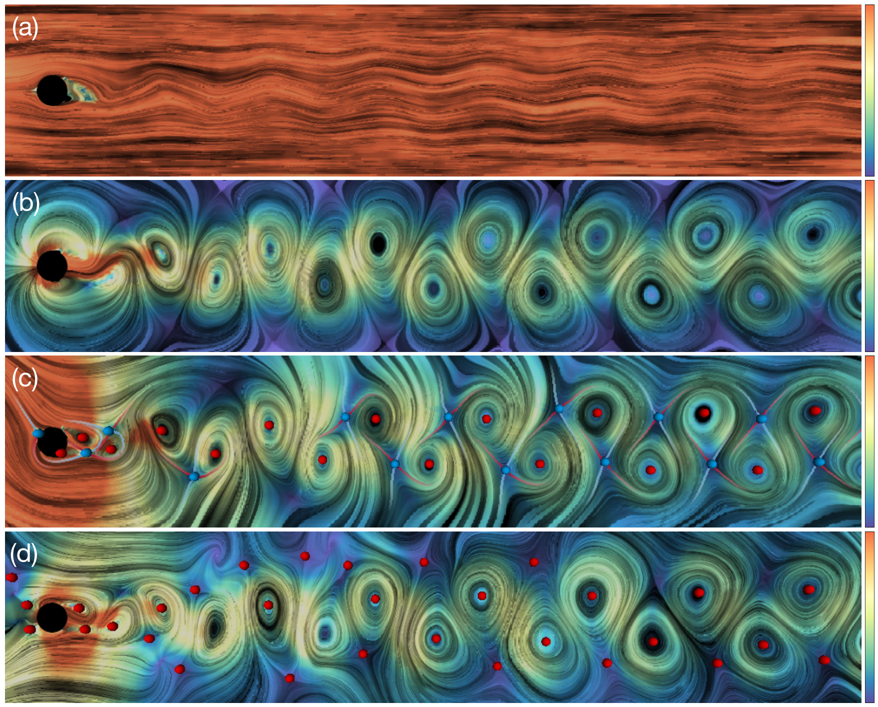

Fig. 1 Visualization of the flow behind a cylinder without (a) and with (b) the background flow removed. For comparison, (c) shows the corresponding Galilean invariant vector field introduced by Bujack et al. constructed from the extrema of the determinant of the Jacobian. The Galilean invariant critical points are marked with red nodes for vortices and with blue nodes for saddles. Image courtesy of Bujack et al. [1]. (d) Galilean invariant vector field introduced in this paper constructed from the extended robustness. The maxima of the extended robustness field are marked with red nodes. (a)-(d): The speed of the flow is color-coded with a rainbow colormap.

based on Galilean invariant criteria, the so-called *Galilean invariant* critical points, and visualizes their frame of reference in their local neighborhood. Here, Galilean invariance refers to the principle that Newton's laws hold in all frames moving at a uniform relative velocity. Thus, a Galilean invariant property is one that does not change when observed in different frames at uniform motion relative to each other. The extrema of the determinant of the Jacobian are shown to be particular examples of such Galilean invariant critical points [1], and they simultaneously capture all relevant features in the data, as illustrated in Fig. 1(c). The intuition is that the determinant of the Jacobian determines the type of critical point, and since the Jacobian is Galilean invariant, its extrema (with magnitude away from zero) correspond to *stable* critical point locations where small perturbations in the field does not change their types. Such Galilean invariant critical points, in general, do not overlap with the classical zeros of the vector fields; however each of them is equipped with a frame of reference in which it becomes a zero of the field. Such a perspective has been shown to be useful in revealing features beyond those obtainable with a single frame of reference (e.g., Fig. 1(c)).

The topological notion of robustness, on the other hand, introduces new ways to think about the *stability* of critical points with respect to perturbations. Robustness, a concept closely related to *topological persistence* [2], quantifies the stability of critical points, and, therefore, assesses their significance with respect to perturbations to the field. Intuitively, the robustness of a critical point is the minimum amount of perturbation necessary to cancel it within a local neighborhood. Robustness, therefore, helps interpreting a vector field in terms of its structural stability. It has been shown to be useful for increasing the visual interpretability of vector fields [10] in terms of feature extraction, tracking [6], and simplification [5, 7, 8].

Contributions. In this paper, we present new and intriguing observations with respect to these two different notions that quantify stable critical points in vector fields, namely, the one based on Jacobian and the one based on robustness. In particular, we address the following questions: *Can we interpret Galilean invariant vector field analysis based on the determinant of the Jacobian via the notion of robustness? What are the relations between these two seemly different notions?*

Our contributions are:

- We extend the definition of robustness by considering every point as a critical point and introduce the notion of the *extended robustness* field by assigning each point in the domain its robustness when it is made critical with a proper frame of reference.
- We prove that the extended robustness satisfies the criterion of Galilean invariance, where the local maxima of the extended robustness field are the Galilean invariant critical points.
- We prove, theoretically, that the determinant of the Jacobian is a lower bound for the extended robustness at the same point.
- We demonstrate, visually, that the extended robustness helps to interpret the Jacobian-based Galilean invariant vector field analysis, in particular, that the extrema of the determinant of the Jacobian coincide with the local maxima of the extended robustness (Fig. 1(c)-(d)).

Technical Background. Here, we discuss some relevant concepts before describing our results.

Galilean invariance. Let $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denote a 2D vector field describing the instantaneous velocity of a flow. A *Galilean transformation* of points $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$ is the composition of a time-dependent translation b that depends linearly on time (i.e., $\dot{b} = const$) and a rigid body rotation $A \in SO(2)$ [1]. A point whose position in the original frame is x then has the coordinate in the transformed frame $x' = Ax + b$ [9]. A vector field $v(x,t)$ is *Galilean invariant* (GI), if it transforms under a *Galilean transformation* A, according to the rule $v'(x',t') = Av(x,t)$ [9]. Similarly, a scalar field $s(x,t)$ and a matrix field $M(x,t)$ are called *GI* if $s'(x',t') = s(x,t)$ and $M'(x',t') = AM(x,t)A^{-1}$, respectively [9].

Reference frame adjustment. Every point in a vector field can be transformed into a critical point by addition of a constant vector. For a time-dependent vector field $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and a point $x_0 \in \mathbb{R}^2$, we define the associated vector field

$v_{x_0} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ with its frame of reference based around x_0 by $v_{x_0}(x,t) := v(x,t) - v(x_0,t)$. Such a vector field v_{x_0} has a critical point at x_0, because $v_{x_0}(x_0,t) = v(x_0,t) - v(x_0,t) = 0$. For a given position $x_0 \in \mathbb{R}^2$, the vector field v_{x_0} is GI, because it is easy to verify that $v'_{x'_0}(x',t') = Av_{x_0}(x,t)$.

Jacobian-based GI vector fields. Recall $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a 2D vector field, where $v(x,t) = \dot{x} = dx/dt = (v_1(x,t), v_2(x,t))^T$. Let J denote the Jacobian of a velocity field,

$$J = \nabla v(x,t) = \begin{pmatrix} \partial v_1(x,t)/\partial x_1 & \partial v_1(x,t)/\partial x_2 \\ \partial v_2(x,t)/\partial x_1 & \partial v_2(x,t)/\partial x_2 \end{pmatrix}$$

The determinant of the Jacobian, $\det(J)$, is shown to be a GI scalar field [1], that is, $\det J'(x',t') = \det J(x,t)$. Such a determinant can be used to categorize first order critical points, that is, a negative determinant corresponds to a saddle, while a positive determinant corresponds to a source, a sink, or a vortex.

A point $(x_0,t_0) \in \mathbb{R}^2 \times \mathbb{R}$ is a Jacobian-based *GI critical point* (GICP) of a vector field $v : \mathbb{R}^2 \times \mathbb{R}^2$ if it is a critical point of the determinant of the Jacobian, i.e. $\nabla \det(J) := \nabla \det(\nabla v(x_0,t_0)) = 0$ [1]. Bujack et. al. [1] restrict such a definition to the negative minima and the positive maxima of the determinant field. The former form saddles, while the latter form sources, sinks, and vortices, respectively, in the velocity field in some specific frame of reference. Each GICP comes with their own frame of reference in which it becomes a classical critical point.

To visualize the GICPs simultaneously, Bujack et. al. [1] have introduced the notion of GI vector field that is applicable beyond Jacobian-based GICPs. The basic idea is to construct a derived vector field that locally assumes the inherent frames of references of each GICP. Such a derived vector field is constructed by subtracting a weighted average of the velocities of the GICPs, x_1,...,x_n, of the vector field v. Formally, let $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be a vector field, $x_1, \ldots, x_n \in \mathbb{R}^2$ a set of GICPs, and w_i the weights of a linear interpolation problem $\sum_{i=1}^n w_i(x)v(x_i)$ with weights $w_i : \mathbb{R}^2 \to \mathbb{R}$ (and a mapping $x \mapsto w_i(x)$) that are invariant under Galilean transformation, that is, $w'_i(x') = w_i(x)$, and the weights add up to one, $\forall x \in \mathbb{R}^2 : \sum_{i=1}^n w_i(x) = 1$. Many commonly used weights satisfy such a condition, for example, constant, barycentric, bilinear, or inverse distance interpolation [1]. Then, the *GI vector field* (GIVF) $\bar{v} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by $\bar{v}(x) := v(x) - \sum_{i=1}^n w_i(x)v(x_i)$. In this paper, we use inverse distance weighting with exponent 2.

Robustness. Let $f, h : \mathbb{R}^2 \to \mathbb{R}^2$ be two continuous 2D vector fields. We define the distance between the two mappings as $\mathrm{d}(f,h) = \sup_{x \in \mathbb{R}^2} ||f(x) - h(x)||_2$. h is an *r-perturbation* of f, if $\mathrm{d}(f,h) \leq r$. Given $f : \mathbb{R}^2 \to \mathbb{R}^2$, the *robustness* of a critical point quantifies the stability of a critical point with respect to perturbations of the vector fields [10]. Intuitively, if a critical point has a robustness value of r, then there exists an $(r+\delta)$-perturbation h of f to eliminate x (via critical point cancellation); and any $(r-\delta)$-perturbation is not enough to eliminate x (see [10] for technical details).

Our Main Theoretical Results. We extend the definition of robustness by considering every point as a critical point. Formally, let $(x_0,t_0) \in \mathbb{R}^2 \times \mathbb{R}$ be an arbitrary point in a vector field $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and $R(x_0,t_0)$ be the robustness of the critical point $(x_0,t_0) \in \mathbb{R}^2 \times \mathbb{R}$ in the vector field v_{x_0}, which is associated with the frame of reference of (x_0,t_0). We define the *extended robustness* $R : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ of the point (x_0,t_0) as the robustness of the critical point $(x_0,t_0) \in \mathbb{R}^2 \times \mathbb{R}$ in the vector field v_{x_0}.

For a vector field $v : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, we call a point a *locally robust critical point* (LRCP) if it is a local maximum in the extended robustness field. The following theorem is the central theoretical contribution of our paper.

Theorem 1 *The extended robustness is a Galilean invariant scalar field. The locally robust critical points defined above are Galilean invariant.*

We prove Theorem 1 by showing that for the extended robustness $R : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, we have $R'(x',t') = R(x,t)$.

Furthermore, we demonstrate that the absolute value of the determinant of the Jacobian is a lower bound on the extended robustness at any critical point.

Theorem 2 *At any point $x_0 \in \mathbb{R}^2$, if the absolute value of the determinant of the Jacobian is at least c, then the extended robustness at x_0 is at least $O(c^2)$.*

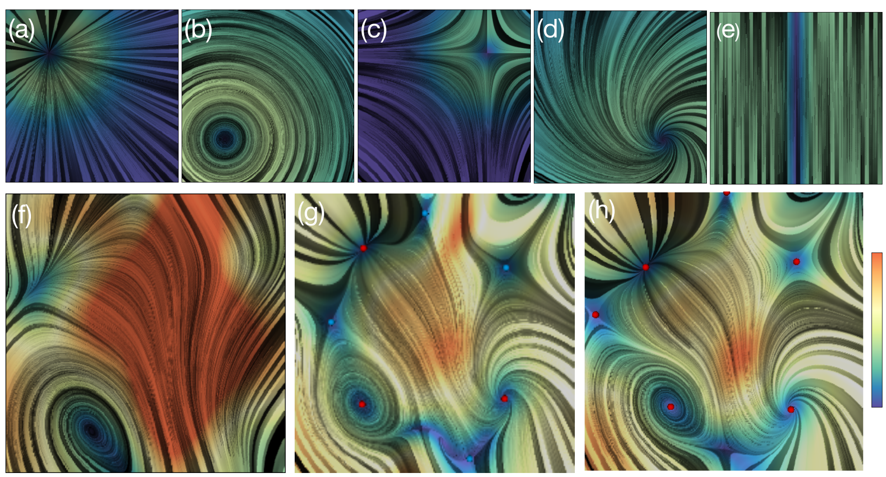

Fig. 2 Visualization of an analytic data set (f), which is created by superimposing five analytic fields (a)-(e). For comparison, (g) shows the corresponding Galilean invariant vector field introduced by Bujack et al. constructed from the extrema of the determinant of the Jacobian. The Galilean invariant critical points are marked with red nodes for vortices and with blue nodes for saddles. Image courtesy of Bujack et al. [1]. (h) the Galilean invariant vector field introduced in this paper constructed from the extended robustness. The maxima of the extended robustness field are marked with red nodes. (a)-(h): The speed of the flow is color-coded with a rainbow colormap.

Our Visualization Results. We would like to demonstrate visually that the extended robustness helps to interpret the Jacobian-based GI vector field analysis, in particular, that the extrema of the determinant of the Jacobian (the Jacobian-based GICPs) coincide with the local maxima of the extended robustness (the LRCPs).

We use an analytic vector field in Fig 2(f). It contains four standard flow features, sink (a), center (b), saddle (c) and spiral source (d), each showing a different common velocity profile overlaid with a sheer flow (e) that makes it impossible to view all the flow features simultaneously. As illustrated, the GIVF based on the determinant of the Jacobian (Fig 2(g)) simultaneously highlights the Jacobian-based GICPs, which correspond to the standard flow features described in Fig 2(a)-(d). On the other hand, these flow features coincide almost perfectly with the features surrounding the LRCPs of the GIVF based on the extended robustness (Fig 2(g)).

Discussion. The Jacobian is the matrix of all first-order partial derivatives of a vector-valued function. It generalizes the usual notion of derivative at a point. In our context, it carries important information about the local behavior of a vector field. On the other hand, robustness leverages relations among critical points to quantify their stability with a Morse theoretical flavor. Its theoretical foundation, the well group theory [3, 4], can be considered as an extension of Morse theory and persistent homology. Therefore, it is probably not entirely surprising that local (differential) behavior of a critical point is linked with topology. In this work, we demonstrate their relations theoretically and visually. Furthermore, our results inspire discussions regarding different quantifiers of stable features within the vector field data.

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